Helsinki Institute of Physics preprint
HIP 1997-02/Th

hep-th/97002132

Quasiclassical Limit in q-Deformed Systems, Noncommutativity and the q-Path Integral

M. Chaichian

High Energy Physics Division, Department of Physics,
University of Helsinki
and
Helsinki Institute of Physics,
P.O.Box 9 (Siltavuorenpenger 20 C),
FIN-00014 Helsinki, Finland

A.P. Demichev

Nuclear Physics Institute,
Moscow State University,
119899, Moscow, Russia

P.P. Kulish

St.Petersburg Branch of
Steklov Mathematical Institute,
Fontanka 27,
191011, St.Petersburg, Russia

Abstract

Different analogs of quasiclassical limit for a q-oscillator which result in different (commutative and non-commutative) algebras of "classical" observables are derived. In particular, this gives the q-deformed Poisson brackets in terms of variables on the quantum planes. We consider the Hamiltonian made of special combination of operators (the analog of even operators in Grassmann algebra) and discuss q-path integrals constructed with the help of contracted "classical" algebras.

1 Introduction

Quasi-classical limit (QCL) in quantum mechanics is very important both from general and utilitarian points of view. First of all, QCL provides connection of classical and quantum theories (see, e.g. [1]) and, in fact, the very definition of quantization process is essentially based on the notion of QCL [2]. On the other hand, in appropriate situations there is effective quasi-classical approximation for calculations in quantum mechanics.

As is well known, quantization can be considered as the (continuous) deformation $\mathcal{O}_\hbar$ of an algebra $\mathcal{O}_0$ of observables (real functions) on classical phase space, so that a deformed associative product (called star-product or $\star$-product) becomes non-commutative [3, 4]. Then QCL appears as a result of expansion of a $\star$-product in powers of the deformation parameter $\hbar$ and in terms of the non-deformed commutative product. In particular, commutators of elements of $\mathcal{O}_\hbar$ (with respect to a $\star$-product) give in first approximation in $\hbar$ the Poisson brackets for elements (functions) of $\mathcal{O}_0$.

A few last years more general class of algebra deformations [5, 6, 7] (see also, e.g. [8, 9] and refs. therein) has attracted great attention. One of the simplest but very important and, in a sense, basic example of deformed algebras is the so called q-oscillator algebra [10, 11, 12, 13]. In an appropriate basis this algebra $\mathcal{B}_q$ is generated by the elements b, b^+ with the commutation relations (CR)

$$bb^+ - q^2 b^+ b = 1 , \qquad q \in \mathbb{R} . \tag{1}$$

Here q is the deformation parameter of a usual oscillator (in this paper we will consider only real values of q). The rescaling of the operators $b \to b/\eta,\ b^+ \to b^+/\eta,\ \ \eta \in \mathbb{R}$ brings to the CR (1) the second free parameter [14] and this CR becomes

$$bb^+ - q^2 b^+ b = \eta^2 , \qquad q \in \mathbb{R} . \tag{2}$$

We shall denote the corresponding algebra as $\mathcal{B}_{q,\eta}$ (all $\mathcal{B}_{q,\eta}$ for $\eta \neq 0$ are isomorphic to $\mathcal{B}_{q,1} \equiv \mathcal{B}_q$). The values

$$q^2 = 1 , \qquad \eta^2 = \hbar ,$$

give the algebra $\mathcal{B}_{1,\hbar}$ of the usual harmonic oscillator with the CR

$$aa^+ - a^+ a = \hbar , \tag{3}$$

(for this specific values of the parameters we use the ordinary notation for oscillator operators: $b, b^+|_{q^2=1,\eta^2=\hbar} = a, a^+$).

The value $\hbar = 0$ gives CR for the coordinates on classical commutative phase space. Transition from (3) to the commutative case corresponds to the usual quasi-classical limit in quantum mechanics.

We are going to consider two other limits for the deformation parameters of $\mathcal{B}_{q,\eta}$; in other words, two other contractions of this algebra:

1. The limit $\eta^2 \to 0,\ q = f(\eta):\ f(\eta) \to 1$ as $\eta \to 0$.
   This leads to *commutative dynamics* in the classical limit. If, e.g. $q^2 = 1+\eta^2\gamma$, the CR (2) becomes (cf. [14])
   $$[b, b^+] = \eta^2(1 + \gamma b^+ b)\ , \tag{4}$$
   and we expect to obtain in this limit classical dynamics with commutative observables, curved phase space and the symplectic form
   $$\omega(\bar{z}, z) = i(1 + \gamma \bar{z} z)^{-1} dz \wedge d\bar{z}\ , \tag{5}$$
   ($\bar{z}, z$ are coordinates on the classical phase space). In this case it is worth to identify the parameter η^2 with the Planck constant $\hbar$.
2. The limit $\eta^2 \to 0$ with fixed q^2. In this case one can expect to obtain q-classical dynamics with *q-commuting variables* (coordinates on a q-plane)
   $$z\bar{z} = q^2 \bar{z} z\ . \tag{6}$$

To study the first limit, we shall use the Bargmann-Fock representation for the q-oscillator algebra and the corresponding symbol calculus in the space with commuting variables but q-deformed differential and integral calculi. Poisson bracket for this case have been introduced earlier [14], in the present paper we shall consider the series expansion in the deformation constant η^2 of the $\star$-product. Of course, the resulting classical phase space is a usual manifold with ordinary differential calculus (for $\gamma < 0$ the phase space proves to be separated into two parts because of the degeneracy of the Poisson brackets related to (5); this is the reason for the appearance of the non-Fock representations for the q-oscillator with $q < 1$, see [23]).

The second limit is more unusual: the contracted algebra is also non-commutative. The CR for the coordinates $\bar{z}, z$ on a q-plane can be considered as the Weyl form of CR for a usual oscillator. Indeed, the operators $\bar{z}, z$ can be represented in the form

$$z = e^{\sqrt{\kappa} a}\ , \qquad \bar{z} = e^{\sqrt{\kappa} a^+}\ , \qquad \kappa \in \mathbb{R}_+\ , \tag{7}$$

where a, a^+ satisfy (3). As follows from the Baker-Hausdorff formula, the operators $\bar{z}, z$ in this case obey (6) with $q^2 = e^{\kappa\hbar}$. Thus the contraction $\eta^2 \to 0$ (as well as the contraction

$q^2 \to 1$) of the q-deformed oscillator algebra leads to the algebra of usual harmonic oscillator but with CR in Weyl form (on the relation between Heisenberg and Weyl commutation relations, see [15]).

The main aim of this paper is to study this limit (contraction to the Weyl algebra) in details. Using differential and integral calculi on Heisenberg-Weyl algebra (i.e. the calculi on a q-plane) [16, 17, 18] we shall develop appropriate q-deformed symbol calculus and find the first approximation (with respect to the parameter η^2) for the product in the algebra of a q-oscillator in terms of Heisenberg-Weyl algebra. Therefore we shall show that the symbol calculus can be applied to study a deformation (at least central one) of not only commutative algebras (e.g., algebras of functions on classical phase spaces) but in the case of deformations of non-commutative algebras as well. The expansions of the product in the algebra $\mathcal{B}_{q,\eta}$ in powers of η^2 allows to define q-deformed analog of a Poisson brackets and thus a kind of a q-deformed "classical" mechanics.

To define integral kernels of operators in terms of variables on a q-plane one needs to consider arbitrary number of copies of "coordinates" on q-planes. In other words, one must define extended algebra of arbitrary number of pairs of operators z and $\bar{z}$ with CR (6) with differential and integral calculi on it. As a byproduct, with help of this extended algebra we shall construct interesting Hamiltonian for q-oscillator: with non-equidistant spectrum but harmonic dependence of the operators $b(t), b^+(t)$ on time t. It is natural to call such a system q-harmonic oscillator. This Hamiltonian belongs to the class of operators which resembles even combinations in Grassmann algebra.

The most important motivation for our consideration is necessity to develop approximation methods for the study of systems with non-canonical (deformed) commutation relations. One possible way to achieve this is connected with q-deformed path integral [16, 18, 19] heavily based on a symbol calculus. In the last Section we shall discuss the meaning of q-deformed path integrals defined in terms of non-commutative variables and its relation with q-deformed "classical" equation of motion.

2 Bargmann-Fock representation for q-oscillator algebra in terms of operators on quantum planes

For the reader convenience we start from recalling of main formulas concerning differential and integral calculi on a q-plane and Bargmann-Fock (BF) representations for the q-oscillator [16, 17, 18, 19].

Two parametric differential calculus invariant with respect to quantum Euclidean

group $E(2)_{r,p}$ is defined by the following commutation relations

$$\begin{array}{cc} z\bar{z} = p\bar{z}z \ , & \partial\bar{\partial} = p\bar{\partial}\partial \ , \\ dzd\bar{z} = -pd\bar{z}dz \ , & (dz)^2 = (d\bar{z})^2 = 0 \ , \\ zd\bar{z} = p(d\bar{z})z \ , & \bar{z}dz = p^{-1}(dz)\bar{z} \ , \\ zdz = r^{-1}(dz)z \ , & \bar{z}d\bar{z} = r(d\bar{z})\bar{z} \ , \\ \partial z = 1 + r^{-1}z\partial \ , & \bar{\partial}\bar{z} = 1 + r\bar{z}\bar{\partial} \ , \\ \partial\bar{z} = p^{-1}\bar{z}\partial \ , & \bar{\partial}z = pz\bar{\partial} \ , \\ dz\bar{\partial} = p^{-1}\bar{\partial}dz \ , & d\bar{z}\partial = p\partial d\bar{z} \ , \\ d\bar{z}\bar{\partial} = r\bar{\partial}d\bar{z} \ , & dz\partial = r^{-1}\partial dz \ . \end{array} \tag{8}$$

In the standard BF representation the derivative $\bar{\partial}$ plays the role of annihilation operator and a multiplication by the coordinate $\bar{z}$ plays the role of creation operator. As is seen from (8), q-deformed $\bar{\partial}$ and $\bar{z}$ satisfy the CR (1) for q-deformed oscillator operators if one puts $r = q^2$. The second parameter p can take arbitrary value: this does not influence the CR (1). On the other hand properties of the quantum plane, which supposedly plays the role of a classical phase space for the q-oscillator, strongly depends on the value of parameter p: if $p = 1$ the plane has commutative coordinates, if $p \neq 1$ the "classical phase space" proves to be non-commutative.

Let us consider the construction of BF representation in two cases:

1. non-commutative plane with $p = r = q^2$;
2. commutative q-plane with $p = 1, \ r = q^2$.

Consider first the non-commutative space, $p = r = q^2$.

To define a scalar product in BF Hilbert space we need a q-analog of integration. In the case of usual non-deformed BF representation the scalar product of two antiholomorphic functions is defined with the help of exponential measure [20]

$$< g, f >= \int d\bar{v}dv \ e^{-\bar{v}v}\overline{g(\bar{v})}f(\bar{v}) \ , \tag{9}$$

($\bar{v}, v$ are commuting variables on ordinary complex plane). In a q-deformed case corresponding integral is defined by the following postulates [16, 17, 18]

1. Normalization
$$I_{00} = \int d\bar{z}dz \ e_{q^2}^{-\bar{z}z} \ = 1 \, ;$$
2. Analog of the Stokes formula
$$\int d\bar{z}dz \ \bar{\partial} f(\bar{z}, z) = \int d\bar{z}dz \ \partial f(\bar{z}, z) = 0 \ ;$$
3. Change of variables
$$\int d\bar{z}dz \ e_{q^2}^{-a^2\bar{z}z} f(\bar{z}, z) = a^{-2} \int d\bar{z}dz \ e_{q^2}^{-\bar{z}z} f(a^{-1}\bar{z}, a^{-1}z) \ .$$

These postulates fix the form of the deformed exponent in the integrand and allow to prove the following result,
$$I_{mn} = \int d\bar{z}dz \ e_{q^2}^{-\bar{z}z} z^n \bar{z}^m = \delta_{mn}[n; q^2]! \ , \qquad n, m = 0, 1, 2, \dots , \tag{10}$$
which is sufficient to compute the integral of any function which has a power series expansion.

The BF representation is constructed in the Hilbert space $\mathcal{H}$ of anti-analytic functions $f(\bar{z})$ on the q-plane with scalar product of the form
$$< g, f >= \int d\bar{z}dz \ e_r^{-\bar{z}z} \overline{g(\bar{z})} f(\bar{z}) \ , \tag{11}$$
so that the monomials
$$\psi^n(\bar{z}) = \frac{\bar{z}^n}{\sqrt{[n; q^2]!}} \ , \qquad [n; q^2] = \frac{q^{2n} - 1}{q^2 - 1} \ , \tag{12}$$
form the orthonormal complete set of vectors in $\mathcal{H}$. The creation and annihilation operators are represented as coordinate and derivative
$$b^+ = \bar{z} \ , \qquad b = \bar{\partial} \ . \tag{13}$$
One can check that b^+ and b are hermitian conjugate of each other with respect to the scalar product (11).

Before going further we notice that a Hilbert space based on operator-valued functions is rather unusual object in quantum mechanics. But from general point of view the appearance of the scalar product (11) and calculations of operator averages with help of it does not mean that we construct essentially new theory in compare with ordinary quantum mechanics. To explain this, it is enough to remember that in general case states of a system are represented in quantum mechanics by density *operators* ρ, so that mean-value $\bar{X}$ of an observable (operator) X is defined by
$$\bar{X} = \mathrm{Tr}(\rho X) \ . \tag{14}$$

On the other hand, the scalar product (11) also can be understood as a trace of the integrand. Indeed, using the representation of the algebra of operators on q-plane in ℓ^2-space [21] spanned by vectors $|k\rangle_W, \;\; k \in \mathbb{Z}$:

$$z|k\rangle_W = q^{-(k+1/2)}|k+1\rangle_W \ , \qquad \bar{z}|k\rangle_W = q^{-(k-1/2)}|k-1\rangle_W \ , \tag{15}$$

one can check that for $q > 1$ trace of the integrand in (10) does exist and after the appropriate normalization coincides with the value of the q-integral (right hand side in (10)) obtained by the pure algebraic axiomatic way. Thus the calculation of operators mean-values with help of the operator valued wave functions (12) again has the form (14) with the integrand in (11) playing the role of a density operator.

Now consider another possibility: plane with commuting coordinates, $p = 1, \; r = q^2$. The BF representation of CR (1) with commuting $z, \bar{z}$ variables and q-deformed differential and integral calculus is constructed in the space of antiholomorphic functions of the form

$$\psi_n = \frac{\bar{z}^n}{\sqrt{[n; q^2]!}} \ ,$$

which are orthonormalized with respect to the scalar product

$$< g, f >= \int d\bar{z}dz \; e_{1/r}^{-q^2\bar{z}z}\overline{g(\bar{z})}f(\bar{z}) \ ,$$

where the integral is defined by the relation

$$\int d\bar{z}dz \; e_{1/r}^{-q^2\bar{z}z} z^n \bar{z}^m = \delta_{mn}[n; q^2]! \ . \tag{16}$$

This integral is defined by the same postulate as in the non-commutative case. Here we have used the so-called second q-deformed (basic) exponential function [22],

$$\exp_{1/q^2}\{x\} = \sum_{n=0}^{\infty} \frac{x^n}{[n; q^{-2}]!} = \sum_{n=0}^{\infty} q^{n(n-1)} \frac{x^n}{[n; q^2]!} \ .$$

This q-deformed integral can be interpreted in terms of Jackson integral, i.e. as sum over sites of a lattice. Thus in this case there is quite ordinary quantum mechanical interpretation of the scalar product and calculations of mean-values of observables.

3 q-Oscillator as the deformation of commutative algebra

As mentioned in Introduction, the adequate formalism to study the quasi-classical limit is the symbol calculus [2, 3, 4]. In the case of oscillator-like systems, the most convenient symbol map is the *normal* one, which put in correspondence to a regular function

$\mathcal{N}_A(\bar z, z) = \sum_{m,n} c_{mn} \bar z^m z^n$ the operator $A = \sum_{m,n} c_{mn}(b^+)^m b^n$, so that in each monomial of the operator, creation operators are on the left of annihilation ones.

In the Bargmann-Fock representations this map can be made explicit using the operator kernels $\mathcal{A}$ of an operator A defined by the relation

$$(Af)(\bar z_1) = \frac{1}{\eta^2} \int d\bar z_2 dz_2 \; \mathcal{A}(\bar z_1, z_2) \mathcal{E}(\bar z_2, z_2) f(\bar z_2) \; ,$$

where the integration is understood in an appropriate sense (usual integration for non-deformed oscillator or the integrations considered in the preceding Section in q-deformed case) and $\mathcal{E}(\bar z, z)$ is the function defined by Bargmann-Fock scalar products, e.g.

$$\begin{aligned} \mathcal{E}(\bar z, z) &= e^{-\bar z z/\hbar} \; , && \text{in non-deformed case;} \\ \mathcal{E}(\bar z, z) &= e_{1/q^2}^{-q^2 \bar z z/\eta^2} \; , && \text{deformed BFR with } p = 1, \; r = q^2 \; ; \\ \mathcal{E}(\bar z, z) &= e_{q^2}^{-\bar z z/\eta^2} \; , && \text{deformed BFR with } p = r = q^2 \; . \end{aligned}$$

Of course, in cases of non-commutative variables one has to define CR between different pairs and care about the order of factors in the integrand. In non-deformed case the kernel of an operator is expressed through the normal symbol by the relation

$$\mathcal{A}(\bar z, z) = e^{\bar z z/\hbar} \mathcal{N}(\bar z, z) \; ,$$

and the product of two operators corresponds to the convolution of their kernels. This allows to find the explicit form of star-product of normal symbols and its quasi-classical limit.

Now let us turn to the q-deformed cases. At first we will consider the algebra $\mathcal{B}_{q,\eta}$ in the limit:

$$\eta^2 \to 0, \qquad q^2(\eta) \to 1$$

(the case 1 in Introduction). As we expect to obtain classical dynamics with commutative observables, we express operators in terms of integral kernels with commuting variables (but q-deformed differential and integral calculi with $p = 1, \; r = q^2$). Because of the same reason it is natural to consider the parameter η^2 as the physical Planck constant and denote it as $\hbar$.

Manipulations analogous to those in the non-deformed case, give that the kernel of product $A_1 A_2$ of two operators A_1, A_2 with kernels $\mathcal{A}_1, \mathcal{A}_2$, equals to the convolution

$$\mathcal{A}_1 * \mathcal{A}_2(\bar z, z) = \frac{1}{\hbar} \int \; d\bar\xi d\xi e_{1/r}^{-q^2 \bar\xi \xi/\hbar} \mathcal{A}_1(\bar z, \xi) \mathcal{A}_2(\bar\xi, z) \; ,$$

constructed with help of the q-integral (16), and the relation between the normal symbol $\mathcal{N}$ and the kernel $\mathcal{A}$ of an operator A is

$$\mathcal{A}(\bar z, z) = e_r^{\bar z z/\hbar} \mathcal{N}(\bar z, z) \; .$$

So to understand how nontrivial (nonzero) $\hbar$ modifies the classical (commutative) multiplication we must calculate in the quasi-classical limit $\hbar \sim 0$ the expression

$$\mathcal{N}_{\mathcal{A}_1} \star \mathcal{N}_{\mathcal{A}_2} = \mathcal{N}_{\mathcal{A}_1 * \mathcal{A}_2} = \frac{1}{\hbar} \int \; d\bar{\xi} d\xi \, \mathcal{N}_1(\bar{z}, \xi) \mathcal{N}_2(\bar{\xi}, z) e_r^{\bar{z}\xi/\hbar} e_{1/r}^{-q^2\bar{\xi}\xi/\hbar} e_r^{\bar{\xi}z/\hbar} e_{1/r}^{-\bar{z}z/\hbar} \; .$$

In the usual classical case of the harmonic oscillator one would combine the exponents, then shift the variables of integration and use, e.g., the steepest descent method to evaluate the integral in $\hbar \to 0$ limit. None of these methods can be used in q-deformed case. Instead, we can use the q-exponent expansion and the q-integral property (10).

Consider operators of the form $P_m = f_m(b^+) b^m$, where $m \in \mathbb{Z}_+$ and f_m is an arbitrary polynomial. Any operator can be represented as a sum (possibly infinite) of P_m. For the convolution of two such operators one has (after trivial change of integration variables $\bar{\xi}, \xi \longrightarrow \bar{\xi}/\sqrt{\hbar}, \xi/\sqrt{\hbar}$)

$$\mathcal{N}_{\mathcal{P}_m * \mathcal{P}_l} = \int d\bar{\xi} d\xi \; f_m(\bar{z}) \xi^m g_l(z) \bar{\xi}^l (\sqrt{\hbar})^{m+l} e_r^{\bar{z}\xi/\sqrt{\hbar}} e_{1/r}^{-q^2\bar{\xi}\xi} e_r^{\bar{\xi}z/\sqrt{\hbar}} e_{1/r}^{-\bar{z}z/\hbar}$$

$$= \sum_{s,n=0}^{\infty} \int d^2\xi f_m(\bar{z}) \xi^m g_l(z) \bar{\xi}^l (\sqrt{\hbar})^{m+l-s-n} \frac{\bar{z}^s \xi^s}{[s;q^2]!} \frac{\bar{\xi}^n z^n}{[n;q^2]!} e_{1/r}^{-q^2\bar{\xi}\xi} e_{1/r}^{-\bar{z}z/\hbar} \; . \tag{17}$$

For definiteness consider the case $l \leq m$ (opposite case is dealt quite similarly). Then from (10) we have

$$\mathcal{N}_{\mathcal{P}_m * \mathcal{P}_l} = f_m(\bar{z}) g_l(z) e_{1/r}^{-\bar{z}z/\hbar} \sum_s \hbar^{l-s} \bar{z}^s z^{m+s-l} \frac{[m+s;q^2]!}{[s;q^2]![s+m-l;q^2]!}$$

$$= f_m(\bar{z}) g_l(z) e_{1/r}^{-\bar{z}z/\hbar} \sum_s \hbar^{l-s} \bar{z}^s z^{m+s-l} \frac{[s+m-(l-1);q^2] ... [s+m;q^2]}{[s;q^2]!} \; . \tag{18}$$

To go further, let us note that the sum of the form

$$\sum_{s=0}^{\infty} \bar{z}^s z^{s-l} \hbar^{l-s} \frac{1}{[s-k;q^2]!} \; ,$$

where $k < l-1$, can be rewritten up to $O(\hbar)$ terms as

$$\begin{aligned} \sum_{s=k}^{\infty} \bar{z}^s z^{s-l} \hbar^{l-s} \frac{1}{[s-k;q^2]!} &= \sum_{s=0}^{\infty} \bar{z}^{s+k} z^{s+k-l} \hbar^{l-k} \hbar^{-s} \frac{1}{[s;q^2]!} \\ &= \bar{z}^k z^{k-l} \hbar^{l-k} e_r^{\bar{z}z/\hbar} \; . \end{aligned} \tag{19}$$

Use of the summation theorem [22] for q-exponentials with commuting arguments

$$e_q^A e_{1/q}^B = \sum_{n=0}^{\infty} \frac{(A+B)_q^{(n)}}{[n;q]!} \; , \qquad AB = BA \; , \tag{20}$$

where

$$(A+B)_q^{(n)} = (A+B)(A+qB)\dots(A+q^{n-1}B) \ ,$$

gives that this sum being inserted in (18) instead of the sum there would produce the expression of the order $\sim O(\hbar^2)$. So in (18) we must take into account the parts of the last factor of the form $\sim 1/[s-l;q^2]!$ and $\sim 1/[s-l+1;q^2]!$ only.

The identities

$$[i+j;q^2] = q^j([i;q^2]-[-j;q^2]) = q^j[i;q^2]+[j;q^2] \ , \tag{21}$$

allow to present the ratio

$$\frac{[s-l+1+m;q^2]\dots[s+m;q^2]}{[s-l+1;q^2]\dots[s;q^2]} \ , \tag{22}$$

in the form

$$\begin{aligned} & q^{ml} + q^l[m;q^2]\frac{1}{[s-l+1;q^2]}\sum_{i=0}^{l-1} q^{-i} \\ & + \ \ \text{terms with higher orders of } [s;q^2] \text{ in denominators,} \end{aligned} \tag{23}$$

so that for (18) we have

$$\begin{aligned} & f_m(\bar z)g_l(z)e_{1/r}^{-\bar z z/\hbar}\sum_s \hbar^{l-s}\bar z^s z^{m+s-l}\frac{[s+m-(l-1);q^2]...[s+m;q^2]}{[s;q^2]!} \\ = \ & f_m(\bar z)g_l(z)e_{1/r}^{-\bar z z/\hbar}\sum_s \hbar^{l-s}\bar z^s z^{m+s-l}\frac{q^{ml}}{[s-l;q^2]!} \\ + \ & f_m(\bar z)g_l(z)e_{1/r}^{-\bar z z/\hbar}q^l[m;q^2] \\ \times \ & \left(\sum_{i=0}^{l-1} q^{-i}\right)\sum_s \hbar^{l-s}\bar z^s z^{m+s-l}\frac{1}{[s-l+1;q^2]!} + O(\hbar^2) \ . \end{aligned} \tag{24}$$

The series over s in this expression can be summed up to $e_r^{\bar z z/\hbar}$, so that we finally obtain for the convolution, up to $\sim O(\hbar)$ terms

$$\mathcal{N}_{\mathcal{P}_m * \mathcal{P}_l} = q^{ml}\mathcal{N}_{\mathcal{P}_m}\mathcal{N}_{\mathcal{P}_l} + \hbar f_m(\bar z)z^{m-1}\bar z^{l-1}g_l(z)q^l[m;q^2]\sum_{i=0}^{l-1}q^{-i} + O(\hbar^2) \ . \tag{25}$$

The factor q^{ml} in the first term shows that in the limit $\hbar \to 0$ the convolution does not reduce to usual commutative multiplication of functions on "classical" phase space. This means that there is no such a classical limit for q-deformed oscillator with $q \neq 1$. From the other hand, if $q = 1 + \gamma\hbar + O(\hbar^2)$, we have for (25) in the same order in $\hbar$

$$\mathcal{N}_{\mathcal{P}_m * \mathcal{P}_l} = \mathcal{N}_{\mathcal{P}_m}\mathcal{N}_{\mathcal{P}_l}(1+ml\gamma\hbar) + \hbar ml f_m(\bar z)z^{m-1}\bar z^{l-1}g_l(z) + O(\hbar^2)$$

$$= \mathcal{N}_{\mathcal{P}_m}\mathcal{N}_{\mathcal{P}_l} + \hbar(1+\gamma\bar{z}z)\partial\mathcal{N}_{\mathcal{P}_m}\bar{\partial}\mathcal{N}_{\mathcal{P}_l} + O(\hbar^2) \ . \qquad (26)$$

Here $\partial, \bar{\partial}$ denote, obviously, the usual (non-deformed) derivatives.

The Poisson bracket derived from (26) has the form

$$\begin{aligned} \{\mathcal{N}_1, \mathcal{N}_2\}_p &= \lim_{\hbar\to 0} \frac{i}{\hbar}\left(\mathcal{N}_1 \star \mathcal{N}_2 - \mathcal{N}_2 \star \mathcal{N}_1\right) \\ &= i(1+\gamma\bar{z}z)\left(\partial\mathcal{N}_1\bar{\partial}\mathcal{N}_2 - \partial\mathcal{N}_2\bar{\partial}\mathcal{N}_1\right) \ , \qquad (27) \end{aligned}$$

and, as we expected, it corresponds to the symplectic form (5).

4 q-Oscillator as the deformation of Weyl algebra

Now consider the second possibility for the quasi-classical limit mentioned in Introduction which leads to non-commutative "classical" variables:

$$\eta^2 \to 0 \ , \quad q \text{ is fixed.}$$

Here we use the q-deformed BF representation with $p = r = q^2$ (see Section 2). Again the action of any operator A in the BF Hilbert space $\mathcal{H}$ based on non-commutative variables, can be represented by its kernel $\mathcal{A}(\bar{z}, z)$

$$(Af)(\bar{z}_1) = \frac{1}{\eta^2}\int d\bar{z}_2 dz_2 \ \mathcal{A}(\bar{z}_1, z_2) e_r^{-\bar{z}_2 z_2/\eta^2} f(\bar{z}_2) \ , \qquad (28)$$

where

$$\mathcal{A}(\bar{z}_1, z_2) = \sum_{m,n} A_{mn} \frac{\bar{z}_1^m}{\sqrt{\eta^{2m}[m;q^2]!}} \frac{z_2^n}{\sqrt{\eta^{2n}[n;q^2]!}} \ , \qquad (29)$$

and we use here the q-integral (10). The special order of operator kernel and "the integration measure" (q-exponent) in (28) is convenient for the subsequent definition of convolution of operators in the case of q-commuting "classical" variables. Another new feature in this definition is that one more pair of q-commuting coordinates is introduced. So we have to define the CR for coordinates on different copies of q-planes. We postulate (cf. [18]) that any copies of coordinates $\bar{z}_i, \ z_i \ (i = 1, 2, ...)$ on q-planes have the following CR:

$$\begin{aligned} z_i\bar{z}_j = q^2\bar{z}_j z_i \ , \qquad & \bar{z}_i\bar{z}_j = \bar{z}_j\bar{z}_i \ , \\ & z_i z_j = z_j z_i \ , \qquad (30) \end{aligned}$$

i.e. they do not depend on the indices which distinguish the copies.

Now we have to derive CR for differentials, derivatives and coordinates for *different* pairs of variables. As usual we can do this using the consistency requirement. For example,

assume that CR for a derivative ∂_z and coordinate ξ from another copy of variables is a homogeneous one: $\partial_z \xi = a\xi\partial_z$, where a is a constant to be defined. Then, acting by both sides of this relation on a function $f(z)$, we obtain $a = 1$. Proceeding in this way, one comes to the following CR for any two different pairs of variables $\{\bar{z}, z\}$ and $\{\bar{\xi}, \xi\}$:

$$
\begin{array}{rclcrcl}
\partial_z \bar{\xi} &=& q^{-2}\bar{\xi}\partial_z \ , & \qquad & \bar{\partial}_z \xi &=& q^2 \xi \bar{\partial}_z \ , \\
\partial_z \xi &=& \xi \partial_z \ , & & \bar{\partial}_z \bar{\xi} &=& \bar{\xi}\bar{\partial}_z \ , \\
dz\bar{\xi} &=& q^2 \bar{\xi} dz \ , & & d\bar{z}\xi &=& q^{-2}\xi d\bar{z} \ , \\
dz\xi &=& \xi dz \ , & & d\bar{z}\bar{\xi} &=& \bar{\xi} d\bar{z} \ , \\
dz d\bar{\xi} &=& q^2 d\bar{\xi} dz \ , & & d\bar{z} d\xi &=& q^{-2} d\xi d\bar{z} \ , \\
dz d\xi &=& d\xi dz \ , & & d\bar{z} d\bar{\xi} &=& d\bar{\xi} d\bar{z} \ , \\
\partial_z \bar{\partial}_\xi &=& q^2 \bar{\partial}_\xi \partial_z \ , & & \bar{\partial}_z \partial_\xi &=& q^{-2}\partial_\xi \bar{\partial}_z \ , \\
\partial_z \partial_\xi &=& \partial_\xi \partial_z \ , & & \bar{\partial}_z \bar{\partial}_\xi &=& \bar{\partial}_\xi \bar{\partial}_z \ , \\
\partial_z d\bar{\xi} &=& q^{-2} d\bar{\xi}\partial_z \ , & & \bar{\partial}_z d\xi &=& q^2 d\xi \bar{\partial}_z \ , \\
\partial_z d\xi &=& d\xi \partial_z \ , & & \bar{\partial}_z d\bar{\xi} &=& d\bar{\xi}\bar{\partial}_z \ .
\end{array}
$$

One can check that the definition of q-integral is consistent with these CR for different variables if one requires that CR of the symbol $\int d\bar{z}dz$ (map from q-plane to $\mathbb{C}$) are defined by (i.e. coincide with) the CR for $d\bar{z}dz$. This gives sense to the notation for the functional ("definite integral") on a q-plane.

Coefficients A_{mn} in (29) can be expressed through the scalar product

$$A_{mn} = q^{2m(n+1)-2m} < \psi_m \mid A \mid \psi_n > \ . \tag{31}$$

Consider an action of two operators A_1 and A_2 on arbitrary wave function $f(\bar{z})$

$$A_2 A_1 = \frac{1}{\eta^2}\int d\bar{z}_1 dz_1 \ \mathcal{A}_2(\bar{z}_2, z_1) e_r^{-\bar{z}_1 z_1/\eta^2} \frac{1}{\eta^2}\int d\bar{z}_0 dz_0 e_r^{-\bar{z}_0 z_0/\eta^2} \mathcal{A}_1(\bar{z}_1, z_0) f(\bar{z}_0) \ . \tag{32}$$

Using the CR for different pairs of q-variables, we obtain the formula for the convolution of operator kernels

$$\mathcal{A}_2 * \mathcal{A}_1(\bar{z}, z) = \frac{1}{\eta^2}\int d\bar{\xi} d\xi \ \mathcal{A}_2(q^{-2}\bar{z}, q^2\xi) e_r^{-\bar{\xi}\xi/\eta^2} \mathcal{A}_1(\bar{\xi}, z) \ . \tag{33}$$

Before going further, let us note that the evolution of an operator in quantum mechanics is defined by its commutator with the Hamiltonian of a system. The use of q-commutators which have the form

$$[(b^+)^m b^n, (b^+)^k b^l]_q \equiv \Big((b^+)^m b^n\Big)\Big((b^+)^k b^l\Big) - q^{2(nk-ml)}\Big((b^+)^k b^l\Big)\Big((b^+)^m b^n\Big) \ ,$$

(the q-factor being chosen accordingly to the homogeneous part of the commutation relations (2)), contradicts the Leibniz rule property of the time derivative in the Heisenberg equation of motion, while the commutation relations for general operators, made of b^+, b, are unnatural and cumbersome. We recall that in somewhat analogous situation for fermionic (anticommuting) operators, the problem is solved by the choice of Hamiltonians from the even subalgebra of the complete fermionic algebra. In our case, the operators which have natural commutation relations with all other operators can be constructed with help of the number operator N due to the equalities

$$bq^{-2N} = q^{-2}q^{-2N}b \ , \qquad b^+q^{-2N} = q^2q^{-2N}b^+ \, . \tag{34}$$

It is easy to check that the monomials of the type

$$q^{-2mN}(b^+)^m b^m \ , \qquad m = 1, 2, ... \tag{35}$$

have natural commutators with any other operators (i.e., the homogeneous parts of the relations have no q-factors). Of course, the number operator N is not independent [23] and in the Fock representation is connected with the creation and annihilation operators by the relation

$$b^+b = \eta^2[N; q^2]; \qquad [N; q^2] \equiv \frac{q^{2N} - 1}{q^2 - 1} \ , \tag{36}$$

that is

$$q^{2N} = 1 + \frac{1}{\eta^2}(q^2 - 1)b^+b \, .$$

Thus, the operators (35) read as

$$\Big(1 + \frac{1}{\eta^2}(q^2 - 1)b^+b\Big)^{-m}(b^+)^m b^m \ . \tag{37}$$

However, there are a few reasons to treat the operator q^{-2N} on a footing distinguished from general operators made of b^+, b: *i)* the relation (36) is correct for the physically most important but still particular Fock representation [23]; *ii)* the operators (37) are non-polynomial ones and, hence, finding of the normal form for them is a quite complicated problem; *iii)* the most essential reason in the context of our consideration is that the commutation relations (34) are the *homogeneous* one and do not depend on the algebra contraction parameter η^2 and on the representation.

The latter fact inspires to introduce the notion of the q-deformed normal symbol. Let us denote for shortness

$$K \equiv q^{-2N} \ ,$$

and define a q-normal monomial

$$N^p_{rs} = K^p(b^+)^r b^s \ , \tag{38}$$

with the corresponding q-deformed normal symbol

$$\mathcal{N}^p_{rs} = K^p \bar{z}^r z^s \ . \tag{39}$$

The crucial difference from the usual normal symbol is that (39) is still an operator expression and that the operator $K \equiv q^{-2N}$ is the same as in the initial operator (38). We imply that the commutation relations of K with $z, \bar{z}$ are the same as with b, b^+

$$zK = q^{-2} K z \ , \qquad \bar{z} K = q^2 K \bar{z} \ . \tag{40}$$

¿From (29) and (31) we can derive the expression for the corresponding integral kernel

$$\mathcal{A}^p_{rs} = q^{2(s(s+1)-r(p+1))} \bar{z}^r z^s \exp_{1/q^2}\{q^{2(s-p+1)} \bar{z} z/\eta^2\} \ . \tag{41}$$

Now we are ready to calculate the normal symbol corresponding to the product of two monomials in K, b^+ and b in quasi-classical approximation in analogy to the case of commuting variables. Consider the convolution of two operator kernels

$$\begin{aligned} \mathcal{A}^p_{ab} * \mathcal{A}^t_{cd} &= \frac{1}{\eta^2} \int d\bar{\xi} d\xi q^{2(b(b+1)-a(p+1))} q^{-2a+2b} \bar{z}^a \xi^b \exp_{1/q^2}\{q^{2(b-p+1)} \bar{z}\xi/\eta^2\} \\ &\times \exp_{q^2}\{-\bar{\xi}\xi/\eta^2\} q^{2(d(d+1)-c(t+1))} \bar{\xi}^c z^d \\ &\times \exp_{1/q^2}\{q^{2(d-t+1)} \bar{\xi} z/\eta^2\} \ . \end{aligned} \tag{42}$$

Long but straightforward calculations of this convolution up to $O(\eta^2)$ terms are analogous to those in the case of commuting variables. However, one has to care about the order of all factors in the expressions. The result is

$$\begin{aligned} \mathcal{A}^p_{ab} * \mathcal{A}^t_{cd} &= q^{(b+d)(b+d+1)-(a+c)(p+t+1)} q^{2bc} q^{2t(a-b)} \bar{z}^{a+c} z^{b+d} \\ &\times \exp_{1/q^2}\{q^{2(b+d-p-t+1)} \bar{z} z/\eta^2\} \\ &+ q^{(b+d-1)(b+d)-(a+c-1)(p+t+1)} q^{2(b-1)(c-1)} q^{2t(a-b)} [b;q^2][c;q^2] \\ &\times \bar{z}^{a+c-1} z^{b+d-1} \exp_{1/q^2}\{q^{2(b+d-p-t)} \bar{z} z/\eta^2\} + O(\eta^4) \ . \end{aligned} \tag{43}$$

This convolution corresponds to the normal symbol

$$\begin{aligned} \mathcal{N}_{\mathcal{A}^p_{ab} * \mathcal{A}^t_{cd}} &= q^{2bc} q^{2t(a-b)} K^{p+t} \bar{z}^{a+c} z^{b+d} + \eta^2 q^{2(b-1)(c-1)} q^{2t(a-b)} [b;q^2][c;q^2] \\ &\times K^{p+t} \bar{z}^{a+c-1} z^{b+d-1} + O(\eta^4) \ . \end{aligned} \tag{44}$$

Introducing the star-product for normal symbols

$$\mathcal{N}^p_{ab} \star \mathcal{N}^t_{cd} = \mathcal{N}_{\mathcal{A}^p_{ab} * \mathcal{A}^t_{cd}} \ , \tag{45}$$

we can present it in the form

$$\mathcal{N}^p_{ab} \star \mathcal{N}^t_{cd} = \mathcal{N}^p_{ab}\mathcal{N}^t_{cd} + \eta^2(\mathcal{N}^p_{ab}\partial_R)(\bar{\partial}\mathcal{N}^t_{cd}) + O(\eta^4) \ , \tag{46}$$

where for convenience we have introduced right derivative ∂_R which has the same CR as the left derivative ∂ but acts on functions from the right. The same concerns its complex conjugate $\bar{\partial}_R$. In particular,

$$z^n\partial_R = [n;q^2]z^{n-1} \ , \qquad \bar{z}^n\bar{\partial}_R = q^{-2(n-1)}[n;q^2]\bar{z}^{n-1} \ .$$

Using (46), one can easily obtain the expression for q-commutator of star-products of normal symbols in quasi-classical limit

$$\begin{aligned} &\left(\mathcal{N}^p_{ab} \star \mathcal{N}^t_{cd} - q^{2(bc-ad+p(d-c)+t(a-b))}\mathcal{N}^t_{cd} \star \mathcal{N}^p_{ab}\right) \\ &= \ \eta^2\mathcal{N}^p_{ab}\left(\partial_R\bar{\partial} - q^{2(b+c-1)}\bar{\partial}_R\partial\right)\mathcal{N}^t_{cd} + O(\eta^4) \ . \end{aligned} \tag{47}$$

An analogy with usual definition and procedure of physical quantization inspires to define a q-deformed "Poisson bracket" as the factor in front of η^2/i in the rhs of (47) in the limit $\eta^2 \to 0$

$$\left\{\mathcal{N}^p_{ab}, \mathcal{N}^t_{cd}\right\}_q = i\mathcal{N}^p_{ab}\left(\partial_R\bar{\partial} - q^{2(b+c-1)}\bar{\partial}_R\partial\right)\mathcal{N}^t_{cd} \ . \tag{48}$$

This means that in the limit $\eta^2 \to 0$, the normal symbol of q-commutator

$$[N^p_{ab}, N^t_{cd}]_q = N^p_{ab}N^t_{cd} - q^{2(bc-ad+p(d-c)+t(a-b))}N^t_{cd}N^p_{ab} \ , \tag{49}$$

of two monomials of the form (38) divided by η^2/i is equal to the q-Poisson bracket

$$\lim_{\eta^2\to 0}\frac{i}{\eta^2}\mathcal{N}_{[.,.]_q} = \{.,.\}_q \ . \tag{50}$$

There is a special class of monomials, namely N^a_{aa}, $\forall a \in \mathbb{Z}$, for which the q-commutator (49) becomes a usual commutator for any second monomials entering the q-commutator

$$[N^a_{aa}, N^t_{cd}]_q = N^a_{aa}N^t_{cd} - N^t_{cd}N^a_{aa} \ , \quad \forall a,b,d,t \ .$$

As we discussed already, if we want to consider the Heisenberg equations of motion for q-oscillator operators with usual time variables and, hence, with time derivative satisfying usual Leibniz rule, we have to use the Hamiltonians (time shift generators) of the type N^a_{aa}, since only the commutator acts on products of operators according to the Leibniz rule. Of course, this is true as we would like to consider the classical limit of the type described above, so that the q-commutator corresponds to the q-Poisson brackets (48). If we would not care about any limit at all, we could consider usual Heisenberg equations (with a commutator in its right hand side) for any operators constructed from b^+, b.

Operators of the type N_{aa}^a very much resemble even elements of a superalgebra, in the sense that the latter have commutation (and not anticommutation) relations with all other elements of the superalgebra.

Thus the natural choice for Hamiltonian in our case is

$$H_{qh} = N_{11}^1 = \omega K b^+ b \;, \qquad \omega \in \mathbb{R} \;. \tag{51}$$

In the q-oscillator basis $|n\rangle$ (see, e.g. [9]) this Hamiltonian is diagonal

$$H_{qh} \mid n\rangle = E_n \mid n\rangle \;,$$

where

$$E_n = \frac{\omega}{q^2}[n; q^{-2}] \;. \tag{52}$$

The corresponding Heisenberg equations of motion prove to be very simple

$$\frac{\partial b}{\partial t} = \frac{i}{\hbar}\left[H_{qh}, b\right] = -\frac{i\eta^2\omega}{\hbar} q^{-2} K b = -\frac{i\eta^2\omega}{\hbar} b K \;, \tag{53}$$

$$\frac{\partial b^+}{\partial t} = \frac{i}{\hbar}\left[H_{qh}, b^+\right] = \frac{i\eta^2\omega}{\hbar} K b^+ = \frac{i\eta^2\omega}{\hbar} q^{-2} b^+ K \;, \tag{54}$$

with the obvious harmonic solution

$$b(t) = e^{-iq^{-2}\eta^2\omega K t/\hbar} b(0) = b(0) e^{-i\eta^2\omega K t/\hbar} \;, \tag{55}$$

$$b^+(t) = e^{i\eta^2\omega K t/\hbar} b^+(0) = b^+(0) e^{iq^{-2}\eta^2\omega K t/\hbar} \;. \tag{56}$$

"q-Quasiclassical" equations of motion, i.e. the Heisenerg equations (53),(54) in the first approximation with respect to the deformation parameter η^2, are defined by the q-deformed Poisson brackets (cf. (50))

$$\frac{\partial z}{\partial t} = \frac{\eta^2}{\hbar}\left\{H_{cl,qh}, z\right\}_q = -i\frac{\eta^2}{\hbar} q^{-2}\omega K z = -i\frac{\eta^2}{\hbar}\omega z K \;, \tag{57}$$

$$\frac{\partial \bar{z}}{\partial t} = \frac{\eta^2}{\hbar}\left\{H_{cl,qh}, \bar{z}\right\}_q = i\frac{\eta^2}{\hbar}\omega K z = i\frac{\eta^2}{\hbar} q^{-2}\omega \bar{z} K \;. \tag{58}$$

with the solutions of the form (55),(56) as in the quantum case. Here we used the q-normal symbol of H_{qh} as classical Hamiltonian

$$H_{cl,qh} = \mathcal{N}_{H_{qh}} = \omega K \bar{z} z \;. \tag{59}$$

If one puts $\eta^2 = \hbar$ (but q is still independent of $\eta^2 = \hbar$; so that $\kappa \sim \hbar^{-1}$ in (7)), the equations (57),(58) prove to be true classical counterparts of (53),(54). In this case there appears the interesting question about existence of classical systems which have to be described by non-commutative algebras. In particular, the possible area of applications might be the theory of stochastic systems (with the parameter q being related to physical parameters which are responsible for the stochastic behaviour of a system).

If the parameter η^2 is independent of $\hbar$ and, hence, $\bar{z}$, z are considered as quantum mechanical operators, we observe the amusing fact that the first order approximation to the q-oscillator with respect to η^2 formally looks like q-generalization of classical mechanics.

5 Discussion

We have developed deformed symbol calculus which allows to express the product in the q-oscillator algebra as the power series in the contraction parameter η^2 and in terms of the product of the contracted algebra, i.e., in our case, commutative or Weyl algebra (algebra of variables on a quantum plane). The latter case is the non-commutative generalization of the usual quasi-classical approximation in quantum mechanics.

In the preceding section we already mentioned some analogy of our consideration with the case of a superalgebra. Here we would like to add that the algebra describing spin-1/2 particles (in fact, algebra of Pauli matrices) also has different "classical" counterparts, either commutative or anticommutative (Grassmann) algebras [24].

Interesting enough that though in general case contracted algebra is also non-commutative, as well as initial one, equations of motion for its operators are defined by a kind of Poisson brackets (q-deformed Poisson brackets) and not by a commutator with a Hamiltonian. However, we have to note that this q-Poisson bracket can not be represented in the coordinate independent form contrary to the usual commutative case. Remind that in the latter case Poisson brackets $\{\cdot,\cdot\}$ for functions F and H on a symplectic manifold $\mathcal{M}$ with a symplectic 2-form Ω can be defined in the coordinate-free form [25]

$$\{F, H\} = \Omega(IdH, IdF) \ , \tag{60}$$

where the map $I:\ T^*\mathcal{M}_x \to T\mathcal{M}_x$ (i.e., the map from 1-forms to vector fields on $\mathcal{M}$) is defined by the relation

$$\Omega(i, I\nu) = \nu(i) \ , \qquad i \in T\mathcal{M}_x \ , \nu \in T^*\mathcal{M}_x \ .$$

One can consider the 2-form $\Omega_q = d\bar{z} \wedge dz$ as the symplectic form on a q-plane but the peculiar form of the q-factor in q-Poisson brackets (48) prevents from a coordinate-free expression of the kind (60).

Another natural question which appears in connection with q-deformed calculus is about possibility of construction and the meaning of q-deformed path integral in terms of contracted but non-commutative algebras. The usual path integral construction is heavily based on notions of a classical phase space. This is one of reasons which emphasize the importance of the consideration in Sections 3 and 4. Making use of the convolution of kernels for infinitesimal evolution operators we have constructed such an integral for both cases of the q-oscillator algebra contractions considered in the present paper [18, 19]. In particular, for the Hamiltonian H_{qh} (see (51)) the evolution operator can be represented in the form

$$U(\bar{z}, z; t'' - t') = \int \left(\prod_t \frac{d\bar{z}(t)dz(t)}{(1 + (1 - q^2)\bar{z}(t)z(t))} \right) e_{1/r}^{q^2\bar{z}(t'')z(t'')} e^{\{iS_q\}} \ , \tag{61}$$

where

$$S_q = \int_{t'}^{t''} \{ i\phi(\bar{z}(t)z(t))\bar{z}(t)\dot{z}(t) - \omega K \bar{z}(t)z(t) \} \, dt \ , \tag{62}$$

$$\phi(\bar{z}(t)z(t)) = \sum_{r=0}^{\infty} \frac{q^{2r}}{(q^2(1-q^2))^{-1} + q^{2r}\bar{z}(t)z(t)} \ ,$$

$\bar{z}(t),\ z(t)$ are different copies of coordinates on q-planes labeled by the time parameter t.

This result shows that even for the Hamiltonian H_{qh} in (51) which has well defined and simple dynamics in q-classical limit, the q-path integral (61) (defined as the convolution of infinitesimal operator kernels) has a complicated structure with nontrivial (non-flat) integration measure and with the "action" (argument of the exponential in the integrand) which does not lead to correct equations of motion (57),(58) through the formal application of usual principle of least action. The obvious reason for this is the absence of the ordinary "paths" in a non-commutative space.

However, we would like to note that this expression for q-deformed path integral does not contradict the equations of motion (57),(58). Indeed, as we pointed out in Section 2, one can understand the "q-integral" over the non-commutative variables $\bar{z}, z$ as the trace of an operator made of only the combination $(\bar{z}z)$. Thus, in fact, the q-path integral (61), or rather its discrete time approximation, can be understood as a sum over values of the operator made of $\bar{z}(t_i)z(t_i)$ at different moments $t_i,\ i = 0, \ldots, N$. The distinction from a path integral over commutative phase space is that now the operators $\bar{z}$ and z do not have common eigenstates and, hence, only one of them or their hermitian combination $\bar{z}z$ can have definite eigenvalues.

Thus in non-commutative case the path integral corresponds to summation over "reduced trajectories" in the space of eigenvalues of maximal set of commuting operators (in our case, just of one operator $\bar{z}z$). This shows, in turn, that we may expect that evolution of only this operator and its functions derived from the equations of motion (57),(58) and from minimizing the "action" S_q (62) (argument of the exponential in (61)) must coincide. It is easy to see that this is indeed the case.

First of all, notice that from the equations of motion (57),(58) it follows that $\bar{z}(t)z(t)$ is an integral of motion (constant in time). On the other hand, let us introduce the operators

$$\rho(t) = \sqrt{\bar{z}(t)}\sqrt{z(t)} \ , \qquad \tau(t) = \sqrt{z(t)}\left(\sqrt{\bar{z}(t)}\right)^{-1} \ ,$$

so that

$$z = \tau\rho \ , \qquad \bar{z} = \frac{1}{\sqrt{q}}\rho\tau^{-1} \ .$$

Thus the "action" S_q takes the form

$$\begin{aligned} S_q &= \int_{t'}^{t''} \left(i\phi(\rho^2(t)/\sqrt{q})\frac{1}{\sqrt{q}}\rho\tau^{-1}(\dot{\tau}\rho + \tau\dot{\rho}) - \frac{\omega K}{\sqrt{q}}\rho^2(t)\right) dt \\ &= \int_{t'}^{t''} \left(\frac{i}{\sqrt{q}}\phi(\rho^2(t)/\sqrt{q})(\rho^2\dot{\nu} + \rho\dot{\rho}) - \frac{\omega K}{\sqrt{q}}\rho^2(t)\right) dt \ , \end{aligned}$$

where $\nu = \log\tau$. Making formal shift $\nu(t) \to \nu(t) + \delta\nu(t)$ one obtains that the condition $\delta S_q = 0$ implies

$$\frac{d}{dt}\rho^2\phi\left(\rho^2(t)/\sqrt{q}\right) = 0 ,$$

and hence $\rho(t) = const$.

Thus the evolution of the operator which defines “trajectories” in the non-commutative case corresponds to the “minimization” of the q-deformed action.

It is worthwhile to compare again this situation with the case of path integral over Grassmann variables. There one also integrates, in fact, over even combination of the variables (of the type $\bar{z}z$, as above). But additional structures, analogous to the function $\phi(\bar{z}z)$, do not appear due to nilpotence property of Grassmann variables.

Generalization of our considerations to the case of multioscillators (with the final aim at a field theory construction) is straightforward; similar treatments for other algebras and as well for the values of q equal to root of unity, remain an interesting problem for further investigations.

Acknowledgments

A.P.D. thanks the Department of Physics and the Research Institute for High Energy Physics, University of Helsinki, where part of this work has been done, for hospitality; his work was partially supported by the INTAS-93-1630-EXT and RFBR-96-02-16413-a grants. The work of P.P.K. was supported by the RFFI grant 96-01-00851.

References

[1] L.D.Landau and E.M.Lifshitz, Quantum Mechanics (Pergamon Press, New York, 1981)

[2] F.A.Berezin, *Comm.Math.Phys.*, **40** (1975) 153.

[3] J.E.Moyal, *Proc. Cambridge Phil. Soc.* **45** (1949) 99.

[4] F.Bayen, M.Flato, C.Fronsdal, A.Lichnerowicz and D.Sternheimer, *Ann. Phys.* **111** (1978) 61; *Ann. Phys.* **111** (1978) 111.

[5] V.G.Drinfel'd, in *Proc.of the International Congress of Mathematicians (Berkley, 1986)* (American Mathematical Society, 1987), p.798.

[6] L.D.Faddeev, N.Yu.Reshetikhin and L.A.Takhtadjan, *Algebra i Analis* **1** (1989) 178 (Transl.: *Leningrad Math. Jour.* **1** (1990) 193).

[7] S.L.Woronowicz, *Comm. Math. Phys.* **111** (1987) 613.

[8] V.Chari and A.Pressley, *A Guide to Quantum Groups*, (Cambridge Univ. Press, Cambridge, 1994)

[9] M.Chaichian and A.Demichev, *Introduction to Quantum Groups* (World Scientific, Singapore, 1996).

[10] A. Macfarlane, *J. Phys.* **A22** (1989) 4581.

[11] L.C. Biedenharn, *J. Phys.* **A22** (1989) L873.

[12] W. Pusz and S.L. Woronowicz, *Rep. Math. Phys.* **27** (1989) 231.

[13] M. Chaichian and P.P. Kulish, *Phys. Lett.* **B234** (1990) 72.

[14] P.P. Kulish and E. Damaskinsky, *J.Phys.* **A23** (1990) L415.

[15] L.D. Faddeev, *Lett.Math.Phys.* **34** (1995) 249.

[16] L. Baulieu and E.G. Floratos, *Phys. Lett.* **B258** (1991) 171.

[17] K.Kowalski and J.Rembielinski, *J. Math. Phys.* **34** (1993) 2153.

[18] M.Chaichian and A.P.Demichev, *Phys. Lett.* **B320** (1994) 273.

[19] M. Chaichian and A.Demichev, in *Proceedings of the Clausthal Symposium on "Non-linear, Dissipative, Irreversible Quantum Systems"*, eds. H.-D. Doebner, V.K. Dobrev and P. Nattermann (World Scientific, Singapore,1995) p.401.

[20] V. Bargmann, *Proc. Nat. Acad. Sci. USA* **2** (1962) 199.

[21] L.L.Vaksman and L.I.Korogodsky, *Dokl. Akad. Nauk SSSR* **304** (1989) 1036.

[22] H.Exton, *q-Hypergeometric functions and applications* (Ellis Horwood, Chichester, 1983).

[23] P.P. Kulish, *Theor. Math. Phys.* **86** (1991) 108.

[24] M.Chaichian and A.P.Demichev, *Phys. Lett.* **A207** (1995) 23.

[25] V.I. Arnold, *Mathematical Methods in Classical Mechanics* (Springer, Berlin, 1978).